B. Math. (Hons.) Ist Year
Analysis I
Instructor - B. Sury
First semestral Exam
November 12, 2018

Attempt ANY FIVE questions among the six.
Maximum Marks 50; each question carries 10 marks.
In case of choices, only the option first attempted will be evaluated.

Q 1. (4+4+2 marks)
(a) For the sequence

$$1.1, -1.01, 1.001, -1.0001, 1.00001, -1.000001, \cdots$$

find the limit inferior, limit superior, infimum and the supremum.
(b) If $\{a_n\}$ is a sequence of positive, real numbers such that the $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, then prove that $\lim_{n\to\infty} a_n^{1/n} = l$.
(c) By considering the sequence $1, a, ab, a^2b, a^2b^2, a^3b^2, a^3b^3, \cdots$ where a, b are distinct positive numbers, show that the converse of (b) is not true.

OR

Q 1. (7+3 marks)
(i) Let a, b be real numbers with $a > 0$. Prove that the infimum of the set $\{an + b/n : n \in \mathbb{N}\}$ equals $a + b$ if $b \leq 2a$ and equals $am + b/m$ when $b > 2a$, where $m = min\{k \in \mathbb{N} : k \geq -1/2 + \sqrt{b/a + 1/4}\}$.
(ii) For any real t, prove that $\lim_{n\to\infty} \frac{t^n}{n!} = 0$.

Q 2. (4+6 marks)
(a) Test the convergence of the series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$.
(b) Let $\{a_n\}$ be a sequence of non-zero real numbers. Assume

$$\lim_{n\to\infty} n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right)$$

exists and is > 1. Prove that $\sum_n a_n$ converges absolutely.

OR

Q 2. (5+5 marks)
(a) If $\sum_{n\geq 1} a_n$ is an absolutely convergent series of real numbers, and σ is a bijection of the set of natural numbers to itself, prove that $\sum_{n\geq 1} a_{\sigma(n)}$ also converges to the same sum.
(b)Prove that the series $\sum_{n\geq 0} \frac{1}{n!+(n+1)!}$ converges to 1.
Hint: Use telescoping to show that $\sum_{n=0}^{N} \frac{1}{n!+(n+1)!} = 1 - \frac{1}{(N+2)!}$.

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Q 3. (2+8 marks).
Let S be a subset of $\mathbb{R}$. Define its interior S^0.
Prove that $(S^0)^c = \overline{S^c}$, where A^c denotes the complement of a set A.

OR

Q 3. (2+8 marks)
Let S be a subset of $\mathbb{R}$. Define its closure $\bar{S}$.
Prove that $(\overline{S})^c = (S^c)^0$.

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Q 4. (6+4 marks)
(a) For the function $f(x) = \frac{1}{e^{1/x}+1}$ defined for $x \neq 0$, determine whether the left hand and right hand limits exist at 0. Draw a rough graph of $f(x)$.
(b) Prove that a uniformly continuous function defined on a bounded subset of $\mathbb{R}$ must be bounded.

OR

Q 4. (5+5 marks)
(a) Prove that there exists no continuous bijection f from $(0, 1)$ to $[0, 1]$.
(b) Prove that the only functions $g : \mathbb{R} \to \mathbb{R}$ satisfying $|g(x)-g(y)| \leq |x-y|^2$ for all x, y are the constant functions.

Q 5. (5+5 marks)
(a) Compute $lim_{x \to 0+} \frac{\log(x)}{x}$.
(b) Prove that the Taylor series of $e^x + e^{-x}$ converges to it for all real x.

OR

Q 5. (5+5 marks)
(a) Compute $\lim_{x \to \pi/2} \frac{tan(x)}{tan(3x)}$.
(b) Let f be a thrice differentiable function such that $f^{(3)}$ is continuous in a neighbourhood of 0. Suppose $f(0) = f'(0) = f"(0) = 0$ and $f^{(3)}(0) \neq 0$. Use Taylor's formula to deduce that f does not have a local extremum at 0.

Q 6. (5+5 marks)
(a) Let $f : [0,1] \to \mathbb{R}$ be thrice differentiable. Suppose $f(0) = f(1) = f'(0) = f'(1) = 0$. Prove that $f^{(3)}(t) = 0$ for some $t \in (0,1)$.
(b) Let f be an infinitely differentiable function defined on $\mathbb{R}$. Suppose $f(1/n) = 0$ for all natural numbers n. Prove that $f^{(k)}(0) = 0$ for all $k \geq 0$.

OR

Q 6. (5+5 marks)
(a) Consider $f(x) = 2x^4 + x^4 \sin(1/x)$ for $x \neq 0$; $f(0) = 0$. Prove that in each interval $(-t, t)$, the derivative f' takes both positive and negative values.
(b) Suppose g is continuous on $[0,2]$ and differentiable on $(0,2)$. If $g(0) = 0$ and $g(1) = g(2) = 1$, prove that there exists $a \in (0,2)$ such that $g'(a) = 1/2$.